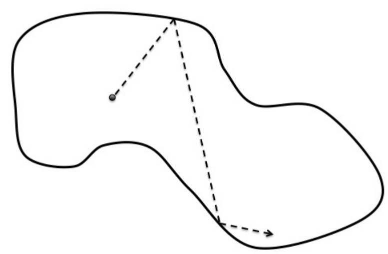
CHEM/CHPH/PHYS 703: Introduction to Nonequilibrium Statistical Physics

PROBLEM SET # 2, due at the start of class, Thursday, March 5, 2020

Problem 1

For an *ergodic* two-dimensional billiard system, show that the mean free path between collisions with the wall is given by the formula $d = \pi A/l$, where A is the area enclosed by the boundary of the billiard, and l is its perimeter.



Hint: Imagine chopping a long trajectory into many very short segments.

Problem 2

A biased coin has probability p to come up heads and $q \equiv 1 - p$ to come up tails. If we set `heads` $=$ `1` and `tails` $=$ `0`, then we can represent the probability distribution of outcomes of this biased coin as a weighted sum of Dirac delta-functions:

$$f(x) = q\,\delta(x) + p\,\delta(x-1) \quad . \tag{1}$$

For N flips of this coin, let $\bar{x} = (1/N)\sum_{i=1}^{N} x_i$ denote the fraction of times that the coin comes up `heads`, and let $f_N(\bar{x})$ be the probability distribution for $\bar{x}$.

(a) Using the binomial distribution, solve for the large deviation function,

$$I(\bar{x}) \equiv \lim_{N\to\infty} -\frac{1}{N} \ln f_N(\bar{x}) \quad . \tag{2}$$

Sketch this function, find its minimum, and evaluate the second derivative at this minimum.

(b) Solve for the generating function, $g(\lambda) = \ln\left\langle e^{\lambda x}\right\rangle$. Confirm that $g'(0)$ and $g''(0)$ are equal to the mean and variance of the distribution $f(x)$.

(c) Solve for the large deviation function $I(\bar{x})$ by taking the Legendre transform of $g(\lambda)$, and confirm that you obtain the same result as for part (a).

Problem 3

A rigid rotor has five degrees of freedom: (x, y, z, θ, ϕ), where the first three specify the location of its center of mass, and the polar and azimuthal angles θ and ϕ specify its orientation. The kinetic energy of a rigid rotor is given by

$$K.E. = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{p_\theta^2}{2I} + \frac{p_\phi^2}{2I\sin^2\theta} \quad , \tag{3}$$

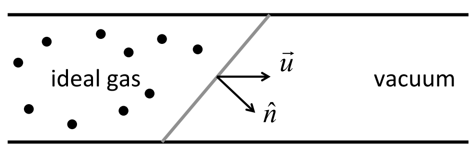
where m is the mass of the rotor and I is its primary moment of inertia.

(a) For a dilute gas of $N \gg 1$ rigid rotors inside a box of volume V, let $\Omega(E, V)$ denote the phase space volume enclosed by the energy shell E. Show that this function has the form

$$\Omega = f_N V^N E^{5N/2} \quad . \tag{4}$$

You do not have to solve explicitly for the quantity f_N.

Problem 4



A piston moves in an infinite cylinder – containing ideal gas on one side and vacuum on the other – with constant velocity $\vec{u}$ parallel to the length of the cylinder. The piston may be "tilted", i.e. the normal vector $\hat{n}$ need not be parallel to $\vec{u}$. The gas begins in thermal equilibrium, and $u \equiv |\vec{u}| \ll \bar{v}$, where $\bar{v}$ is the mean particle speed in the gas.

(a) Assuming first that $\hat{n} \parallel \vec{u}$, show that the energy of the gas changes at a rate

$$\dot{E} = -u\rho\sigma k_B T + mu^2\rho\sigma\bar{v} + \mathcal{O}(u^3) \tag{5}$$

where ρ and T are the number density and temperature of the gas, m is the mass of a gas particle, and σ is the surface area of the piston. How is this formula modified if the piston moves into (rather than away from) the gas? How is it modified if $\hat{n}$ is not parallel to $\vec{u}$?

(b) Use these results to compute the effective friction coefficient for a hard sphere of radius R moving slowly through an ideal gas.